

# SUMS OF HERMITIAN SQUARES AS AN APPROACH TO THE BMV CONJECTURE

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**ABSTRACT.** Lieb and Seiringer stated in their reformulation of the Bessis-Moussa-Villani conjecture that all coefficients of the polynomial  $p(t) = \text{tr}[(A + B)^m]$  are nonnegative whenever  $A$  and  $B$  are any two positive semidefinite matrices of the same size. We will show that for all  $m \in \mathbb{N}$  the coefficient of  $t^4$  in  $p(t)$  is nonnegative, using a connection to sums of Hermitian squares of non-commutative polynomials which has been established by Klep and Schweighofer. This implies by a famous result of Hillar that the coefficients of  $t^k$  are nonnegative for  $0 \leq k \leq 4$ .

## 1. INTRODUCTION

The Bessis-Moussa-Villani (BMV) conjecture, originally stated as a problem of quantum statistical mechanics, has a 30 year long history. Since its introduction in 1975 [1] many partial results have been given, see e.g. [13] for a review until 2000. The following reformulation of Lieb and Seiringer [12] is more capable to algebraic methods than the original one.

**Conjecture 1.1 ((Bessis, Moussa, Villani)).** *For all positive semidefinite matrices  $A$  and  $B$  and all  $m \in \mathbb{N}$ , the polynomial  $p(t) := \text{tr}((A + tB)^m) \in \mathbb{R}[t]$  has only nonnegative coefficients.*

The coefficient of  $t^k$  in  $p(t)$  for a given  $m$  is the trace of  $S_{m,k}(A, B)$ , where  $S_{m,k}(A, B)$  is the sum of all words of length  $m$  in the letters  $A$  and  $B$  in which  $B$  appears exactly  $k$  times. For example  $S_{4,2}(A, B) = A^2B^2 + ABAB + AB^2A + BABA + B^2A^2 + BA^2B$ .

In [1] it has already been shown that the BMV conjecture is true for  $2 \times 2$  matrices. Since for  $0 \leq k \leq 2$  or  $m - 2 \leq k \leq m$  each word in  $S_{m,k}(A, B)$  has nonnegative trace, as is easily seen, the conjecture is true for  $m \leq 5$ . Hillar and Johnson [7] verified the first nontrivial case  $m = 6, k = 3$  for positive semidefinite  $3 \times 3$  matrices. Hägele [4] verified  $m = 7$  which leads by a result of Hillar [6] to  $m \leq 7$ . Further, Klep and Schweighofer [9] derived that Conjecture 1.1 is true for  $m \leq 13$ . Whereas all these results fix  $m$  and consider arbitrary  $k \leq m$ , we take the opposite viewpoint, fix  $k = 4$  and let  $m \in \mathbb{N}$  be arbitrary. We will give a proof that

$$\text{tr}(S_{m,4}(A, B)) \geq 0$$

with no restrictions on  $m$  or the matrix size of  $A$  and  $B$ .

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A result of Hillar [6] then implies that it is true for all  $k \leq 4$  and arbitrary  $m$ , in particular if  $k = 3$  which can't be shown directly by our method.

Using analytical methods, Fleischhack and Friedland [3] showed that for fixed positive semidefinite  $A, B$  and fixed  $k$ , the trace of  $S_{m,k}(A, B)$  is nonnegative whenever  $m$  is big enough. Unfortunately, their lower bound on  $m$  is dependent of  $A$  and  $B$ . Otherwise this would imply the BMV conjecture.

To verify Conjecture 1.1 it is sufficient to show the nonnegativity of  $\text{tr}(S_{m,k}(A, B))$  for any two positive semidefinite *real* matrices  $A$  and  $B$  of the same size. Since further every positive semidefinite real matrix  $A$  is decomposable as  $A = C^2$  for some real matrix  $C$  we specify our examination to  $S_{m,k}(C^2, D^2)$  where  $C$  and  $D$  are any two real symmetric matrices of the same size. To work in an algebraic context we identify  $S_{m,k}(C^2, D^2)$  as a polynomial  $S_{m,k}(X^2, Y^2)$  in two non-commuting variables  $X$  and  $Y$ .

For this let  $\mathbb{R}\langle X, Y \rangle$  denote the unital associative  $\mathbb{R}$ -algebra freely generated by  $X$  and  $Y$ . The elements of  $\mathbb{R}\langle X, Y \rangle$  are polynomials in the non-commuting variables  $X$  and  $Y$  with coefficients in  $\mathbb{R}$ . An element  $w$  of the monoid  $\langle X, Y \rangle$ , freely generated by  $X, Y$ , is called a *word* and  $w_{(i)}$  its  $i$ -th letter. An element of the form  $aw$ , where  $0 \neq a \in \mathbb{R}$  and  $w \in \langle X, Y \rangle$ , is called a *monomial* and  $a$  its *coefficient*. We endow  $\mathbb{R}\langle X, Y \rangle$  with the involution  $p \mapsto p^*$  fixing  $\mathbb{R} \cup \{X, Y\}$  pointwise. In particular, for each word  $w \in \langle X, Y \rangle$ ,  $w^*$  is its reverse. If  $w^* = w$ ,  $w$  is called a *palindrome*. An element of the form  $g^*g$  for some  $g \in \mathbb{R}\langle X, Y \rangle$  is called a *hermitian square*.

Using this terminology we define the polynomial  $S_{m,k}(X, Y)$  as the polynomial in the variables  $X$  and  $Y$  as the sum of all monic monomials of total degree  $m$  and degree  $k$  in  $Y$ . Replacing  $X$  and  $Y$  by  $X^2$  and  $Y^2$  leads to the desired polynomial  $S_{m,k}(X^2, Y^2)$ , which results in  $S_{m,k}(A, B)$  when we evaluate at symmetric matrices  $C$  and  $D$ , satisfying  $C^2 = A$  and  $D^2 = B$ .

The invariance of the trace under cyclic permutations motivates the definition of cyclic equivalence [9]. A cyclic permutation of a word  $v$  of length  $m$  is a map  $\sigma$ , where  $\sigma(v) = v_{(\sigma(1))}v_{(\sigma(2))} \cdots v_{(\sigma(m))}$ , for which there exists some  $k \in \mathbb{N}$  such that  $\sigma(i) = i + k \pmod{m}$  for all  $i = 1, \dots, m$ . For example  $v_{(1)}v_{(2)}v_{(3)} \mapsto v_{(3)}v_{(1)}v_{(2)}$  is a cyclic permutation whereas  $v_{(1)}v_{(2)}v_{(3)} \mapsto v_{(3)}v_{(2)}v_{(1)}$  is not.

**Definition 1.2.** Two words  $v, w \in \langle X, Y \rangle$  are called *cyclically equivalent* ( $v \stackrel{\text{cyc}}{\sim} w$ ) if  $\sigma(v) = w$  for some cyclic permutation  $\sigma$  of  $v$ .

Two polynomials  $f = \sum_w a_w w$  and  $g = \sum_w b_w w$  with  $a_w, b_w \in \mathbb{R}$  are cyclically equivalent if for each  $v \in \langle X, Y \rangle$  the sums of coefficients of all words  $w \in \langle X, Y \rangle$  which are cyclically equivalent to  $v$  are equal, i.e.,  $\sum_{w \stackrel{\text{cyc}}{\sim} v} a_w = \sum_{w \stackrel{\text{cyc}}{\sim} v} b_w$ . This is equivalent to  $f - g$  being a sum of commutators in  $\mathbb{R}\langle X, Y \rangle$ , where the commutator  $[p, q]$  is defined by  $[p, q] := pq - qp$ .

The polynomials  $f = X^2YX + YX^3 + 2X^2Y^2$  and  $g = 2YX^3 + 2YX^2Y$  are cyclically equivalent since  $f - g = [X^2, YX] + [2X^2Y, Y]$ . Alternatively, the condition on the coefficients is easily checked as well.

**Definition 1.3.** The *order* ( $\text{ord } w$ ) of a word  $w = w_{(1)} \cdots w_{(m)}$  of length  $m$  is the smallest positive integer  $k$ , such that  $w_{(i+k)} = w_{(i)}$  for all  $i = 1, \dots, m$  where we identify  $w_{(i+k)}$  with  $w_{(i+k-m)}$  if  $i + k > m$ . Thus cyclically equivalent words have the same order. It can also be defined as the smallest integer  $k \geq 1$  such that there exists a subword  $v = v_{(1)} \cdots v_{(k)}$  of length  $k$  with  $w = v \cdots v = v^{m/k}$ . The

equivalence of these two definitions follows easily by induction over the length of the subword  $v$ .

**Remark 1.4.** One obtains that the order of a word  $w = v^{m/\text{ord}(w)}$  in  $S_{m,4}(X^2, Y^2)$  divides  $m$ . Further, since  $Y^2$  appears the same number of times in every subword  $v$ , we get that  $\frac{m}{\text{ord}(w)}$  divides 4. In particular  $\text{ord}(w) \in \{m, \frac{m}{2}, \frac{m}{4}\} \cap \mathbb{N}$ .

Our main result is the following.

**Theorem 1.5.** *For  $k = 4$  and  $m \in \mathbb{N}$  the polynomial  $S_{m,4}(X^2, Y^2)$  is cyclically equivalent to a sum of Hermitian squares.*

- Remark 1.6.** (i) Hägele [4] has shown that  $S_{6,3}(X^2, Y^2)$  cannot be cyclically equivalent to a sum of Hermitian squares of a certain special form. Landweber and Speer generalized this result to  $k = 3$  and  $m \geq 6$  but  $m \neq 11$  [10]. Using this result a fact of Klep and Schweighofer [9, Prop. 3.1] shows that  $S_{m,3}(X^2, Y^2)$  cannot be cyclically equivalent to any sum of Hermitian squares if  $m \geq 6$  and  $m \neq 11$ . Therefore we are interested in the case  $k = 4$  and arbitrary  $m \in \mathbb{N}$ .
- (ii) Since a sum of Hermitian squares is positive semidefinite on all real symmetric matrices, Theorem 1.5 implies that  $\text{tr}(S_{m,4}(A, B))$ , the coefficient of  $t^4$  in  $p(t)$ , for all  $m \in \mathbb{N}$  is nonnegative for all positive semidefinite matrices  $A, B$ .

In the sequel we will present a proof of Theorem 1.5 by constructing a sum of Hermitian squares which is cyclically equivalent to  $S_{m,4}(X^2, Y^2)$ . By Remark 1.4 the order of words in  $S_{m,4}(X^2, Y^2)$  divides  $m$  and 4. Thus, if  $m$  is odd all words in  $S_{m,4}(X^2, Y^2)$  have order  $m$ , whereas in the even case order  $\frac{m}{2}$  and  $\frac{m}{4}$  are also possible. Therefore we split the proof in two parts,  $m$  odd and even, starting with the easier part, where  $m$  is odd.

## 2. CASE $m$ ODD

To verify Theorem 1.5 it suffices to construct a sum of Hermitian squares  $f$  which is cyclically equivalent to  $S_{m,4}(X^2, Y^2)$ . Let  $m$  be fixed. Since  $S_{m,4}(X^2, Y^2)$  is homogeneous in  $X$  and  $Y$ , one can reduce the set of words in a decomposition as sum of Hermitian squares, as in the commutative case, to the set of words of half the degree in  $X$  and  $Y$ . Thus we set

$$V = \{v \in \langle X, Y \rangle \mid \deg_X v = m - 4, \deg_Y v = 4\}.$$

Further we define the subsets

$$V_0 = \{v \in \{X^2, Y^2\}^{\frac{m-1}{2}} X \mid v = X^k Y^2 X^\ell Y^2 X^{k'+1}, k \leq k'\} \cap V,$$

$$V_1 = \{v \in X \{X^2, Y^2\}^{\frac{m-1}{2}} \mid v = X^{k+1} Y^2 X^\ell Y^2 X^{k'}, k+1 \leq k'\} \cap V.$$

We denote the possible exponents of  $X$  in a word  $v_i$  by  $k_i, \ell_i$  and  $k'_i$  such that for example every  $v_i \in V_0$  is of the form  $v_i = X^{k_i} Y^2 X^{\ell_i} Y^2 X^{k'_i+1}$  and satisfies the condition  $k_i + \ell_i + k'_i = m - 5$  where  $\ell_i, k_i, k'_i \in 2\mathbb{N}$  and  $k_i \leq k'_i$ .

The exponent  $k_i$  (respectively  $k_i + 1$  if  $v_i \in V_1$ ) is bounded by  $d$ , the highest possible even (respectively odd) number which is less than or equal to  $\frac{m-5}{2}$ , thus the maximum of these bounds is in any case  $\frac{m-5}{2}$ .

Now, we will construct a sum of Hermitian squares  $f$ . For given  $k \in \mathbb{N}$  let  $k(2)$  denote the remainder of  $k$  modulo 2. Then we group the words  $v_i \in V_0$  (respectively

$V_1$ ) according to  $k_i$ . For every  $k = 0, 1, 2, \dots, \frac{m-5}{2}$  we add all words  $v_i \in V_{k(2)}$  with  $k_i + k(2) = k$  and obtain a polynomial  $f_k$ . By construction all words in  $f_k^* f_k$  have even exponents in  $X$  and  $Y$ . Finally, we set

$$(1) \quad f := m \sum_{k=0}^{\frac{m-5}{2}} f_k^* f_k.$$

**Example 2.1.** (a)  $m = 7$ : We have  $V_0 = \{Y^2 X^2 Y^2 X, Y^4 X^3\}$  and  $V_1 = \{XY^4 X^2\}$  which leads to

$$f_0 = Y^2 X^2 Y^2 X + Y^4 X^3 \quad \text{and} \quad f_1 = XY^4 X^2$$

and finally

$$\begin{aligned} S_{7,4}(X^2, Y^2) &\stackrel{\text{cyc}}{\sim} 7(f_0^* f_0 + f_1^* f_1) \\ &= 7(XY^2 X^2 Y^4 X^2 Y^2 X + XY^2 X^2 Y^6 X^3 + X^3 Y^6 X^2 Y^2 X + X^3 Y^8 X^3 \\ &\quad + XY^4 X^4 Y^4 X). \end{aligned}$$

This representation is of the same kind as the one given by Hägele in [4].

(b)  $m = 9$ : Since  $V_0 = \{Y^2 X^2 Y^2 X^3, Y^4 X^5, X^2 Y^4 X^3, Y^2 X^4 Y^2 X\}$  and  $V_1 = \{XY^4 X^4, XY^2 X^2 Y^2 X^2\}$  we get by construction

$$\begin{aligned} f_0 &= Y^2 X^2 Y^2 X^3 + Y^4 X^5 + Y^2 X^4 Y^2 X, \\ f_1 &= XY^2 X^2 Y^2 X^2 + XY^4 X^4 \quad \text{and} \\ f_2 &= X^2 Y^4 X^3. \end{aligned}$$

One easily checks  $S_{9,4}(X^2, Y^2) \stackrel{\text{cyc}}{\sim} 9(f_0^* f_0 + f_1^* f_1 + f_2^* f_2)$ .

We will prove that  $f$  is the desired sum of Hermitian squares in two steps. First all words appearing in  $f$  will be shown to be pairwise cyclically inequivalent. By construction each word in  $f$  appears in  $S_{m,4}(X^2, Y^2)$  and has order  $m$ . Since up to cyclic equivalence each word in  $S_{m,4}(X^2, Y^2)$  appears  $m$  times, it suffices to show that the sums of coefficients in both polynomials are the same.

**Remark 2.2.** To compare two words appearing in  $f$  with respect to cyclic equivalence we use the following method. Since  $Y^2$  appears exactly four times in each word  $w$  of  $f$ , we know  $w = X^{n_0} Y^2 X^{n_1} Y^2 X^{n_2} Y^2 X^{n_3} Y^2 X^{n'_4}$  for some  $n_0, \dots, n_3, n'_4$ . Further  $w$  is cyclically equivalent to  $\tilde{w} := Y^2 X^{n_1} Y^2 X^{n_2} Y^2 X^{n_3} Y^2 X^{n_4}$  where  $n_4 = n'_4 + n_0$ , i.e.  $\tilde{w}$  consists of four groups  $Y^2 X^{n_i}$ . Let  $w'$  be another word with exponents  $m_i$ , i.e.,  $\tilde{w}' := Y^2 X^{m_1} Y^2 X^{m_2} Y^2 X^{m_3} Y^2 X^{m_4}$ . Then  $\tilde{w}$  and  $\tilde{w}'$  are the same or  $n_i = m_{i-j} \pmod{4}$  for  $i = 1, \dots, 4$  and  $j = 1, 2, 3$ , which can be obtained by “rotating”  $\tilde{w}'$   $j$  times, i.e., for  $j = 1$  one shifts the first group  $Y^2 X^{m_1}$  to the end, for  $j = 2$  one shifts also the second group to the end and so on, thus  $m_i$  becomes  $m_{i-j}$ .

For simplicity we use the fact that rotating three times is the same as rotating once in the reverse direction, i.e., shifting the group  $Y^2 X^{m_4}$  to the beginning. Thus rotating  $w'$  three times is the same as fixing  $w'$  and rotating  $w$  once. Therefore we can omit  $j = 3$  by symmetry.

**Lemma 2.3.** *All words appearing in  $f$  are pairwise cyclically inequivalent.*

*Proof.* By construction a word  $w$  in  $f$  is either a word in  $\sum_{2k} f_{2k}^* f_{2k}$  thus of the form  $w = v_1^* v_2$  where  $v_1, v_2 \in V_0$  and  $k_1 = k_2$ , i.e.,

$$w = X^{k'_1+1} Y^2 X^{\ell_1} Y^2 X^{2k_1} Y^2 X^{\ell_2} Y^2 X^{k'_2+1} \stackrel{\text{cyc}}{\sim} Y^2 X^{\ell_1} Y^2 X^{2k_1} Y^2 X^{\ell_2} Y^2 X^{k'_1+k'_2+2}.$$

Or it is a word in  $\sum_{2k} f_{2k+1}^* f_{2k+1}$  thus of the form  $w = v_1^* v_2$  where  $v_1, v_2 \in V_1$  and  $k_1 = k_2$ . The same is true for any other word  $w' = v_3^* v_4$ . As is easily seen  $\tilde{w} = \tilde{w}'$  is only possible if  $v_1 = v_3$  and  $v_2 = v_4$ . We are left with the following cases.

If  $w$  and  $w'$  are words in  $\sum_{2k} f_{2k}^* f_{2k}$  which are cyclically equivalent then we have to consider

- (a)  $\ell_1 = 2k_3, \quad 2k_1 = \ell_4, \quad \ell_2 = k'_3 + k'_4 + 2, \quad k'_1 + k'_2 + 2 = \ell_3$  or
- (b)  $\ell_1 = \ell_4, \quad 2k_1 = k'_3 + k'_4 + 2, \quad \ell_2 = \ell_3, \quad k'_1 + k'_2 + 2 = 2k_3$ .

In (a)  $2k_3 + k_1 + k'_1 = \ell_1 + k_1 + k'_1 = \ell_3 + k_3 + k'_3 = k'_1 + k'_2 + 2 + k_3 + k'_3$  leads to  $k_1 + k_3 = k'_2 + k'_3 + 2$  contradicting  $k_1 + k_3 \leq k'_2 + k'_3 < k'_2 + k'_3 + 2$ . Subcase (b) leads to  $2k_1 = k'_3 + k'_4 + 2 > 2k_3 = k'_1 + k'_2 + 2 > 2k_1$ , which is not possible.

The case that  $w, w'$  are words in  $\sum_{2k} f_{2k+1}^* f_{2k+1}$  works the same way.

If  $w$  is a word in  $\sum_{2k} f_{2k}^* f_{2k}$  and  $w'$  a word in  $\sum_{2k} f_{2k+1}^* f_{2k+1}$ , then we have

- (a)  $\ell_1 = k'_3 + k'_4, \quad 2k_1 = \ell_3, \quad \ell_2 = 2k_3 + 2, \quad k'_1 + k'_2 + 2 = \ell_4$  or
- (b)  $\ell_1 = \ell_3, \quad 2k_1 = 2k'_3 + 2, \quad \ell_2 = \ell_4, \quad k'_1 + k'_2 + 2 = k'_3 + k'_4$ .

In (a)  $k'_3 + k'_4 + k_1 + k'_1 = \ell_1 + k_1 + k'_1 = \ell_3 + k_3 + k'_3 = 2k_1 + k_3 + k'_3$  leads to  $k'_1 + k'_4 = k_1 + k_3 = k_1 + k_4 < k'_1 + k'_4$ . Subcase (b) contradicts  $k_1, k'_3 \in 2\mathbb{N}$ .

If  $w$  is a word in  $\sum_{2k} f_{2k+1}^* f_{2k+1}$  and  $w'$  in  $\sum_{2k} f_{2k}^* f_{2k}$ , we exchange  $w$  and  $w'$ .

Summarizing, despite the trivial case that  $w$  and  $w'$  are constructed by the same subwords  $v_i$ , they cannot be cyclically equivalent.  $\square$

Thus every word in  $f$  has its order  $m$  as coefficient. Since up to cyclic equivalence this is the same in  $S_{m,4}(X^2, Y^2)$ , we are done by the following lemma.

**Lemma 2.4.** *The number of pairwise cyclically inequivalent words in  $f$  is the same as in  $S_{m,4}(X^2, Y^2)$ .*

*Proof.*  $S_{m,4}(X^2, Y^2)$  contains  $\binom{m}{4}$  words. Since each word has order  $m$ , there are

$$\frac{1}{m} \binom{m}{4} = \frac{1}{6} \left( \frac{m-3}{2} \right) \left( \frac{m-1}{2} \right) (m-2)$$

pairwise cyclically inequivalent words in  $S_{m,4}(X^2, Y^2)$ .

Let  $k \in \mathbb{N}$  be fixed. Then  $f_k$  consists of  $\frac{m-3}{2} - k$  different words. For example, if  $k$  is even then there are  $\frac{1}{2}(m-5-k_1) + 1$  possibilities for  $k_1, \ell_1, k'_1 \in 2\mathbb{N}$  with  $\ell_1 + k'_1 = m-5-k_1$  (namely  $k'_1 = m-5-k_1-\ell_1, \ell_1 = 0, 2, \dots, m-5-k_1$ ), the restriction  $k_1 \leq k'_1$  of  $V_0$  excludes  $\frac{k_1}{2}$  possibilities.

Thus the number of words in  $f$  is given by

$$\sum_{k=0}^{\frac{m-5}{2}} \left( \frac{m-3}{2} - k \right)^2 = \sum_{k=0}^{\frac{m-3}{2}} k^2 = \frac{1}{6} \left( \frac{m-3}{2} \right) \left( \frac{m-1}{2} \right) (m-2).$$

$\square$

**Remark 2.5.** After we had finished the proof of this case, we heard of the recent work of Landweber and Speer [10] who proved the same result (for odd  $m$ ) by quite

similar techniques; but they haven't investigated the case where  $m$  is even. They found a sum of Hermitian squares which only consists of words  $w$  in

$$V_2 := \{v \in X\{X^2, Y^2\}^{\frac{m-1}{2}} \mid v = X^{k+1}Y^2X^\ell Y^2X^{k'}\} \cap V.$$

Let  $v_i = w_i X \in V_0$  for  $i = 1, 2$ . Starting with  $f$  and using

$$v_1^* v_2 = (Xw_1)^* Xw_2 = w_1^* X X w_2 \stackrel{\text{cyc}}{\sim} Xw_2 w_1^* X = (w_2^* X)^* (w_1^* X) = (v_2^*)^* (v_1^*)$$

and  $V_1 \subseteq V_2$  leads to a sum of Hermitian squares  $\tilde{f}$  which is exactly the representation found by Landweber and Speer.

This result agrees with the more general Proposition 3.1 in [9] which in particular states that independent of  $k$  in the case  $m$  odd once one has found a representation as sum of Hermitian squares one can also find a representation using only of words of  $V_2$ .

### 3. CASE $m$ EVEN

Since words in  $S_{m,4}(X^2, Y^2)$  now have order  $m, \frac{m}{2}$  or  $\frac{m}{4}$ , the constructed polynomial  $f$  of the last section is further not cyclically equivalent to  $S_{m,4}(X^2, Y^2)$ . Thus we will add weights on the words in our construction to respect the different orders.

Let  $m$  be fixed and  $V = \{v \in \langle X, Y \rangle \mid \deg_X v = m - 4, \deg_Y v = 4\}$ . Further we define the subsets

$$V_0 = \{v \in \{X^2, Y^2\}^{\frac{m}{2}} \mid v = X^k Y^2 X^\ell Y^2 X^{k'}, k \leq k'\} \cap V,$$

$$V_1 = \{v \in X\{X^2, Y^2\}^{\frac{m-2}{2}} X \mid v = X^{k+1} Y^2 X^\ell Y^2 X^{k'+1}, k \leq k'\} \cap V.$$

To distinguish even and odd exponents, we define  $\hat{k}_i := k_i + 1$  and  $\hat{k}'_i := k'_i + 1$ . Then every  $v_i \in V_0$  is of the form  $v_i = X^{k_i} Y^2 X^{\ell_i} Y^2 X^{k'_i+1}$  and satisfies  $k_i + \ell_i + k'_i = m - 4$  where  $\ell_i, k_i, k'_i \in 2\mathbb{N}$  and  $k_i \leq k'_i$ , whereas every  $v_i \in V_1$  satisfies  $\hat{k}_i + \ell_i + \hat{k}'_i = m - 4$ . Thus the maximal possible exponent  $k_i$  respectively  $\hat{k}_i$  (if  $m$  is not divisible by 4) is given by  $\frac{m-4}{2}$ .

Now we construct our desired sum of Hermitian squares as follows. Let  $k \in \mathbb{N}$  and let  $k(2)$  denote the remainder of  $k$  modulo 2. For every  $k = 0, 1, 2, \dots, \frac{m-4}{2}$  we add all words  $v_i \in V_{k(2)}$  with  $k_i + k(2) = k$  as in the case where  $m$  is odd, but we weight the words with  $k_i < k'_i$  with coefficient 1 and the words with  $k_i = k'_i$  with coefficient  $\frac{1}{2}$ . This leads to a polynomial  $f_k$  which contains exactly one word with coefficient  $\frac{1}{2}$  whereas all other coefficients are 1. Finally we set

$$(2) \quad f := m \sum_{k=0}^{\frac{m-4}{2}} f_k^* f_k.$$

**Example 3.1.**  $m = 8$ : We have  $V_0 = \{Y^2 X^2 Y^2 X^2, Y^4 X^4, X^2 Y^4 X^2, Y^2 X^4 Y^2\}$  and  $V_1 = \{XY^4 X^3, XY^2 X^2 Y^2 X\}$  which leads to

$$f_0 = Y^2 X^2 Y^2 X^2 + Y^4 X^4 + \frac{1}{2} Y^2 X^4 Y^2$$

$$f_1 = XY^4 X^3 + \frac{1}{2} XY^2 X^2 Y^2 X \quad \text{and}$$

$$f_2 = \frac{1}{2} X^2 Y^4 X^2.$$

Then one easily verifies  $S_{8,4}(X^2, Y^2) \stackrel{\text{cyc}}{\sim} 8(f_0^* f_0 + f_1^* f_1 + f_2^* f_2)$ .

Now for example the words  $w = (Y^2 X^2 Y^2 X^2)^*(Y^2 X^4 Y^2)$  in  $f_0^* f_0$  and  $w' = (XY^2 X^2 Y^2 X)^*(XY^4 X^3)$  in  $f_1^* f_1$  are cyclically equivalent but due to our weights their coefficients sum up to  $\text{ord}(w) = 8$ .

The proof of cyclic equivalence works similarly as in the case where  $m$  is odd. But since there are now cyclically equivalent words appearing in  $f$ , we have to calculate more carefully. We will show first that the sum of coefficients of cyclically equivalent words in  $f$  is less than or equal to their order. Since each word in  $f$  appears in  $S_{m,4}(X^2, Y^2)$  we will finish by showing that the sums of coefficients are equal in both representations.

**Lemma 3.2.** *The sum of coefficients of cyclically equivalent words in  $f$  is less than or equal to the order of the corresponding words.*

*Proof.* We will use the same method as explained in Remark 2.2 of the last section. Let  $w, w'$  be two different words appearing in  $f$  and  $w \stackrel{\text{cyc}}{\sim} w'$ .

If  $w$  and  $w'$  are words in  $\sum f_{2k}^* f_{2k}$ . Then either  $w$  and  $w'$  are equal or one of the following subcases holds:

- (a)  $\ell_1 = 2k_3, \quad 2k_1 = \ell_4, \quad \ell_2 = k'_3 + k'_4, \quad k'_1 + k'_2 = \ell_3$
- (b)  $\ell_1 = \ell_4, \quad 2k_1 = k'_3 + k'_4, \quad \ell_2 = \ell_3, \quad k'_1 + k'_2 = 2k_3$

In subcase (a) we obtain  $k_3 + k_1 = k'_2 + k'_3$  from  $2k_3 + k_1 + k'_1 = \ell_1 + k_1 + k'_1 = \ell_3 + k_3 + k'_3 = k'_1 + k'_2 + k_3 + k'_3$ , thus  $k_1 = k'_2$  and  $k_3 = k'_3$ . Further we obtain from  $\ell_1 + k_1 + k'_1 = \ell_4 + k_3 + k'_4$  that  $k'_1 - k_1 = k'_4 - k_3$ . In (b) we obtain  $2k_1 = k'_3 + k'_4 \geq 2k_3 = k'_1 + k'_2 \geq 2k_1$ , thus equality holds, which leads to  $w = w'$ .

The other cases work in the same way by replacing  $k_i$  by  $k'_i$  whenever  $w$  is a word in  $\sum f_{2k+1}^* f_{2k+1}$  and  $k'_i$  respectively if  $w'$  is a word in  $\sum f_{2k+1}^* f_{2k+1}$ . If  $w$  and  $w'$  are not in the same set  $\sum f_{2k}^* f_{2k}$  or  $\sum f_{2k+1}^* f_{2k+1}$ , then they obviously cannot be equal.

Summarizing, we derive that when  $w \stackrel{\text{cyc}}{\sim} w'$  but  $w \neq w'$  then  $k_1 = k'_2, k_3 = k'_3$  and  $k'_1 - k_1 = k'_4 - k_3$  or by symmetry (confer Remark 2.2)  $k_3 = k'_4, k_1 = k'_1$  and  $k'_3 - k_3 = k'_2 - k_1$  holds, where the first set of equations describes the words which differ by one rotation, and the second set describes the case of three rotations.

Assuming, there are two different words  $w', w''$  both cyclically equivalent to  $w$ . Then all three are pairwise cyclically equivalent and at least two of them (for example  $w', w''$ ) are in  $\sum_k f_k^* f_k$  ( $k$  even or odd). Thus each of them satisfies one set of equations, but then  $w'$  and  $w''$  differ by two rotations, which leads to equality (subcase (b)). Therefore there are at most two words in  $f$  which are pairwise cyclically equivalent.

To conclude the proof, if  $w = v_1^* v_2$  with  $k_1 = k'_1 = k_2 = \frac{m-4}{4}$  then  $\ell_1 = m - 4 - 2k_1 = \frac{m-4}{2} = \ell_2$ , thus  $w$  has order  $\frac{m}{4}$  which is equal to the coefficient of  $w$  in  $f$ . A cyclically equivalent word  $w' = v_3^* v_4$  has to satisfy  $k_3 = k'_3 = k'_4$  and  $2k_3 = \ell_1 = 2k_1$  which leads to  $w = w'$ . Therefore there is no other word  $w' \stackrel{\text{cyc}}{\sim} w$  in  $f$ . In all other cases the coefficient of  $w$  is half of the order of  $w$ . Since there are at most two pairwise cyclically equivalent words we are done.  $\square$

**Lemma 3.3.** *The sum of coefficients in both polynomials is the same.*

*Proof.* The sum of coefficients in  $S_{m,4}(X^2, Y^2)$  is  $\binom{m}{4} = \frac{1}{24}m(m-1)(m-2)(m-3)$ .

For every  $k = 0, 1, 2, \dots, \frac{m-4}{2}$  each polynomial  $f_k$  has one word with coefficient  $\frac{1}{2}$  and  $\frac{m-4}{2} - k$  times coefficient 1. Thus the sum of coefficients in  $f$  is given by

$$\begin{aligned} m \sum_{k=0}^{\frac{m-4}{2}} \left( \frac{m-4}{2} - k + \frac{1}{2} \right)^2 &= \frac{m(m-2)}{8} + m \sum_{k=0}^{\frac{m-4}{2}} (k^2 + k) \\ &= \frac{m}{24} (3(m-2) + (m-4)(m-2)m) = \frac{1}{24} m(m-1)(m-2)(m-3). \end{aligned}$$

□

#### 4. CONCLUDING REMARKS

- (a) To get an idea how sums of Hermitian squares which are cyclically equivalent to  $S_{m,4}(X^2, Y^2)$  might look like, we used numerical computations extending those done by Klep and Schweighofer [9]. In particular we used NCAIgebra [8], YALMIP [11] and SeDuMi [14] as the starting point of our investigation.
- (b) As in the case  $m$  odd one might consider  $V_2 = \{v \in \{X^2, Y^2\}^{\frac{m}{2}}\} \cap V$  if  $m$  is even. Then one can find a much more complicated sum of Hermitian squares which is cyclically equivalent to  $S_{m,4}(X^2, Y^2)$  and consists just of words in  $V_2$  if  $m(4) = 2$ , i.e.,  $m$  is even but not divisible by 4.

Since all words are in the letters  $X^2$  and  $Y^2$  one obtains by substitution that  $S_{m,4}(X, Y)$  is cyclically equivalent to a sum of Hermitian squares, which implies  $\text{tr}(S_{m,4}(A, B)) \geq 0$  for *all* real Hermitian matrices  $A, B$  of the same size. This result has recently, independently been found by Collins, Dykema and Torres-Ayala [2].

- (c) Landweber and Speer [10] showed that despite a few exceptions (which all have been solved) one cannot find a representation of  $S_{m,k}(X^2, Y^2)$  as a sum of Hermitian squares if  $m$  or  $k$  is odd. But they have no negative results if  $m$  and  $k$  are both even. This gap has recently been filled by Collins, Dykema and Torres-Ayala [2] who proved that, despite the case  $(16, 8)$ ,  $S_{m,k}(X^2, Y^2)$  is not cyclically equivalent to a sum of Hermitian squares if  $m - 6 \geq k \geq 6$  and  $m \geq 16$ . Thus this approach cannot proof the BMV conjecture.

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